An Overview on Randomized Algorithms for Analysis and Control of Uncertain Systems*

Roberto Tempo and Fabrizio Dabbene
IEIIT-CNR
Politecnico di Torino – Italy
e-mail: {tempo, dabbene}@polito.it

Abstract

It is well-known that the dominating paradigm of the robustness research is the so-called *worst-case model*. In contrast to this model, the main feature of the approach studied in this paper is to provide a probabilistic assessment on the system robustness.

We present here an overview of this research area, usually denoted as "probabilistic methods for analysis and control of uncertain systems."

1 Introduction

The modern approach for robustness of control systems is mainly focused on a deterministic description of the uncertainty. Along this line of research, fundamental results have been obtained in the last twenty years, see for instance [6],[9],[40] and references therein. The main drawback of this approach, however, lies in the computational complexity limitations arising in a number of problems, as pointed out in several recent papers on NP-hardness in system and control, see e.g. [7],[8],[28]. This issue forces the control engineer to introduce simplifications and overbounding in the description of the uncertainty, in order to obtain a relaxed problem that is easier to deal with. In this way, one trades tightness in the problem description with computational efficiency.

The study of probabilistic methods for analysis and design of control systems has recently received a growing interest in the scientific community. Probabilistic and randomized techniques provide complementary methodologies for studying robustness, with a trade-off between computational complexity and tightness of the solution [13],[15],[31],[41]. The starting point of the randomized approach

*A version of this paper has been published in *Perspectives in Robust Control - Lecture Notes in Control and Information Science*, (ed. S.O. Moheimani), Springer-Verlag.

Paper presented at the RTO SCI Lecture Series on "Robust Integrated Control System Design Methods for 21st Century Military Applications", held in Forlì, Italy, 12-13 May 2003; Setúbal, Portugal, 15-16 May 2003; Los Angeles, USA, 29-30 May 2003, and published in RTO-EN-SCI-142.

is to introduce a probabilistic description of the unknown-but-bounded uncertainty. Unlike classical worst-case methods, randomized algorithms provide a probabilistic assessment on the satisfaction of design specifications. Besides their low complexity, a fundamental advantage of these algorithms resides in the fact that the obtained robustness margins are larger than the classical ones, if a prespecified level of risk is accepted [12]. Probabilistic algorithms can therefore be used by the control engineer, together with classical robustness methods, in order to obtain useful additional information on the problem.

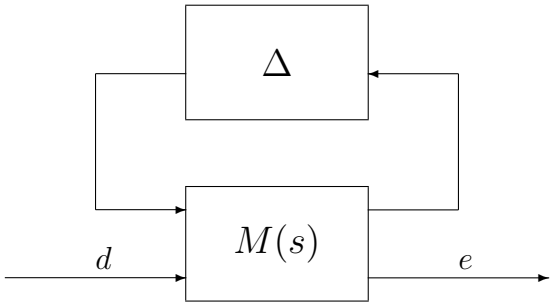

Figure 1: M-Δ configuration.

In this paper, which is a revised and updated version of [38], we present a review of recent results on the probabilistic approach for robustness analysis and design of uncertain systems. In Section 2, we introduce definitions and notation. In Section 3, we derive explicit bounds for the number of samples required to estimate, within a given level of accuracy and confidence, the probability that a control system subject to random uncertainty Δ, with given probability density function (pdf) over a set $\mathbf{\Delta}_\rho$, attains a specified performance level γ. It can be easily shown that the number N of randomly generated samples is independent of the number of blocks (either real, complex or mixed) of the matrix Δ and the size of the set $\mathbf{\Delta}_\rho$. This fact is an immediate consequence of the Law of Large Numbers and is often used in Monte Carlo simulations, see [22],[36],[37]. Subsequently, in Section 4, the problem of sample generation is outlined, and algorithms for the generation of random samples uniformly distributed in various norm-bounded sets are discussed. In Section 5, we show how probabilistic robust design can be performed in this framework. In particular, we discuss iterative algorithms for the design of Linear Quadratic Regulators that guarantee robustness properties in terms of probability. Conclusions are briefly summarized in Section 6.

2 Preliminaries

In this paper, we use the standard M-Δ configuration adopted in worst-case robustness methods; see e.g. [40] for a more general discussion on this topic. In Figure 1, $M(s)$ represents the deterministic part of the system, which consists of

the nominal plant and controller transfer matrices and weighting functions, while Δ is used to describe various perturbations affecting the control system. Disturbances and errors are denoted by d and e, respectively. The class of allowable perturbations is defined as

$$\mathbf{\Delta} \doteq \{\text{blockdiag}\,[q_1 I_{r_1}, \ldots, q_\ell I_{r_\ell}, \Delta_1, \ldots, \Delta_b]\}.$$

The vector $q \doteq [q_1, q_2, \ldots, q_\ell]^T \in \mathbb{F}^\ell$ takes into account real or complex parametric uncertainties affecting the plant, while the matrices $\Delta_i \in \mathbb{F}^{n_i, m_i}$, $i = 1, \ldots, b$ are full block matrices generally introduced to represent high order unmodeled dynamics. The structured matrix Δ is restricted within a set $\mathbf{\Delta}_\rho$ described in terms of norm-bounded balls of radius ρ

$$\mathbf{\Delta}_\rho \doteq \{\Delta \in \mathbf{\Delta} : \|q\| \leq \rho, \|\Delta_i\| \leq \rho, i = 1, \ldots, b\}\,.$$

In the standard μ-theory setting [18],[35], the induced ℓ_2-norm

$$\|\Delta\|_2 \doteq \sup_{\|x\|_2 = 1} \|\Delta x\|_2$$

is used, even though definitions of μ based on the Frobenius-norm have been also studied [23]. The choice of the ℓ_2-norm is clearly equivalent to take $\|q\|_1$ and $\|\Delta_i\|_2$, $i = 1, \ldots, b$. We remark that, depending on the specific problem under attention, different ℓ_p norms may be chosen to describe parametric uncertainty. For example, one can choose ℓ_2 and ℓ_1 norm balls as bounding sets for the uncertainty vector q; see e.g. [3].

In the classical problem of robustness analysis of the control system shown in Figure 1, the goal is to guarantee that a certain performance requirement is attained for all $\Delta \in \mathbf{\Delta}_\rho$. In general, this requirement can be stated in terms of the *performance function*

$$u \doteq u(\Delta),$$

where $u(\Delta)$ is a Lebesgue measurable function of Δ. Without loss of generality, in this paper we study a single performance function, but in general the simultaneous attainment of several performance requirements may be handled with the techniques presented here.

A classical example of performance function deals with the $\mathcal{H}_1$ norm of a stable transfer function matrix $T_{de}(M, \Delta)$ between the disturbances d and the error e. That is, we set

$$u(\Delta) = \|T_{de}(M, \Delta)\|_1\,,$$

and check, for a given performance level γ, if $u(\Delta) < \gamma$ for all $\Delta \in \mathbf{\Delta}_\rho$.

Another example of performance function is related to robust stability of a continuous time MIMO system. To illustrate, consider a state-space realization of M, i.e.

$$M(s) = C(sI - A)^{-1}B + D,$$

with $A \in \mathbb{R}^{n,n}$ stable and B, C, D real matrices of suitable dimensions. Then, if the well-posedness condition on D holds [33], we obtain

$$\{\Delta : A + B\Delta(I - D\Delta)^{i\,1}C \text{ stable}\} = \{\Delta : \det(I - \Delta M(j\omega)) \neq 0, \forall \omega \in \mathbb{R}\}.$$

The performance function $u(\Delta)$ for robust stability of the M-Δ configuration is therefore given by

$$u(\Delta) = \max(\text{Re } \lambda_1(\Delta), \text{Re } \lambda_2(\Delta), \ldots, \text{Re } \lambda_n(\Delta))$$

where $\lambda_1(\Delta), \ldots, \lambda_n(\Delta)$ are the eigenvalues of the uncertain matrix A D

can be taken as a measure of the system robustness. We observe that in classical robustness analysis, one of the objectives is to compute the value of γ such that performance is guaranteed for all $\Delta \in \mathbf{\Delta}_\rho$. This is equivalent to require that the set $\mathbf{\Delta}_{good}$ coincides with $\mathbf{\Delta}_\rho$. On the other hand, in a probabilistic setting, we are satisfied if these two sets "approximately" coincide and, in particular, the ratio

$$\frac{\text{vol}(\mathbf{\Delta}_{good})}{\text{vol}(\mathbf{\Delta}_\rho)} \tag{1}$$

is close to one.

Next, we assume that the structured matrix Δ is a random matrix with associated probability density function $f_\Delta(\Delta)$ and support $\mathbf{\Delta}_\rho$. Formally, for a given set $\mathbf{S} \subseteq \mathbf{\Delta}_\rho$, we define

$$\text{Prob}\{\Delta \in \mathbf{S}\} = \int_{\mathbf{S}} f_\Delta(\Delta) d\Delta.$$

The probabilistic robustness of the M-Δ system is therefore stated in terms of the probability that the system satisfies the desired perfomance. In other words, if Δ is a random matrix, we aim to compute the "weighted" volume of the set $\mathbf{\Delta}_{good}$ with respect to the density function $f_\Delta(\Delta)$

$$\text{Prob}\{\Delta \in \mathbf{\Delta}_{good}\} = \int_{\mathbf{\Delta}_{good}} f_\Delta(\Delta) d\Delta = \text{Prob}\ \{u(\Delta) \leq \gamma\}.$$

Clearly, if $f_\Delta(\Delta)$ is the uniform density on $\mathbf{\Delta}_\rho$, the probability $\text{Prob}\{\Delta \in \mathbf{\Delta}_{good}\}$ coincides with the ratio, defined in (1), of the volumes $\text{vol}(\mathbf{\Delta}_{good})$ and $\text{vol}(\mathbf{\Delta}_\rho)$. We remark that the choice of uniform distribution is common in this setting for its worst-case properties in a certain class of distribution functions [4].

3.1 Performance Verification Problem

We now study the probabilistic version of the performance verification problem. Given a performance level $\gamma > 0$, we aim to estimate the probability of performance

$$p_\gamma \doteq \text{Prob}\ \{u(\Delta) \leq \gamma\}. \tag{2}$$

The exact computation of this probability is in general very difficult, and only in very special cases can be computed in closed form, see e.g. [19]. More generally, p_γ may be estimated by means of a randomized algorithm. To this end, we generate N independent identically distributed (i.i.d.) samples within $\mathbf{\Delta}_\rho$

$$\Delta^1, \Delta^2, \ldots, \Delta^N \in \mathbf{\Delta}_\rho$$

according to the given density function $f_\Delta(\Delta)$. Subsequently, we compute

$$u(\Delta^1), u(\Delta^2), \ldots, u(\Delta^N)$$

and we construct the indicator function

$$I(\Delta^i) \doteq \begin{cases} 1 & \text{if } u(\Delta^i) \leq \gamma; \\ 0 & \text{otherwise.} \end{cases}$$

An estimate of p_γ is immediately obtained as

$$\hat{p}_N = \frac{1}{N} \sum_{i=1}^{N} I(\Delta^i)$$

which is equivalent to

$$\hat{p}_N = \frac{N_{good}}{N}$$

where N_{good} is the number of samples such that $u(\Delta^i) \leq \gamma$. The estimate $\hat{p}_N$ is usually referred to as *empirical probability*.

Clearly, for a finite sample size, it is important to know how many samples are needed to obtain a reliable estimate $\hat{p}_N$ of p_γ. This reliability can be measured in terms of the "closeness" of $\hat{p}_N$ to the probability p_γ. That is, given $\epsilon \in (0,1)$, we impose

$$|p_\gamma - \hat{p}_N| = |\text{Prob}\{u(\Delta) \leq \gamma\} - \hat{p}_N| \leq \epsilon.$$

Since $\hat{p}_N$ is estimated via random sampling, we notice that it is a random variable. Therefore, for given $\delta \in (0,1)$, we require

$$\text{Prob}\{|p_\gamma - \hat{p}_N| \leq \epsilon\} \geq 1 - \delta. \tag{3}$$

The problem is then finding the minimal N such that (3) is satisfied for fixed *accuracy* $\epsilon \in (0,1)$ and *confidence* $\delta \in (0,1)$. An immediate solution of this problem is given by the Bernoulli Law of Large Numbers [5].

Bernoulli Law of Large Numbers
For any $\epsilon \in (0,1)$ *and* $\delta \in (0,1)$, *if*

$$N \geq \frac{1}{4\epsilon^2\delta}$$

then

$$\text{Prob}\{|p_\gamma - \hat{p}_N| \leq \epsilon\} \geq 1 - \delta.$$

The derivation of this bound is very simple and it is based upon the Chebychev inequality [29]. We observe that the number of samples computed with the Law of Large Numbers is independent of the number of blocks of Δ, the size of $\mathbf{\Delta}_\rho$ and the density function $f_\Delta(\Delta)$. Unfortunately, the number of samples N may be very large. For example, if $\epsilon = 0.1\%$ and $1 - \delta = 99.9\%$, we obtain $N = 2.5 \cdot 10^8$. We remark, however, that the cost associated with the evaluation of $u(\Delta^i)$ for fixed Δ^i is polynomial-time in many cases. This is true when dealing, for example, with the computation of $\mathcal{H}_1$ norms or when stability tests are of concern. Therefore,

on the contrary of the worst-case robustness approach (see the discussion in Section 1), we conclude that the total cost to perform probabilistic performance analysis is polynomial-time, provided that polynomial-time algorithms for sample generation are available. This latter problem is addressed in Section 3.

A bound which improves upon the previous one is the Chernoff Bound [16].

Chernoff Bound
For any $\epsilon \in (0,1)$ *and* $\delta \in (0,1)$*, if*

$$N \geq \frac{\log \frac{2}{\delta}}{2\epsilon^2}$$

then

$$\text{Prob}\{|p_\gamma - \hat{p}_N| \leq \epsilon\} \geq 1 - \delta.$$

We remark that the Chernoff Bound largely improves upon the bound of Bernoulli. For example, if $\epsilon = 0.1\%$ and $1-\delta = 99.9\%$, we compute $N = 3.9 \cdot 10^6$.

Finally, we observe that these bounds can be computed *a priori* and are *explicit*. That is, given ϵ and δ one can find directly the minimum value of N. On the other hand, when computing the classical lower and upper confidence intervals, the sample size obtained is not explicit. More precisely, for given $\delta \in (0,1)$, the so-called lower and upper confidence intervals p_L and p_U are such that

$$\text{Prob}\{p_L \leq p_\gamma \leq p_U\} = 1 - \delta.$$

The evaluation of this probability requires the solution with respect to p_L and p_U of equations of the type

$$\sum_{k=N_{good}}^{N} \binom{N}{k} {p_L}^k (1-p_L)^{N\,i\,k} = \delta_L;$$

$$\sum_{k=0}^{N_{good}} \binom{N}{k} {p_U}^k (1-p_U)^{N\,i\,k} = \delta_U,$$

with $\delta_L + \delta_U = \delta$. Clearly, the probabilities p_L and p_U can be computed only *a posteriori*, once the value of N_{good} is known. Moreover, since an explicit solution of the previous equations is not available, standard tables are generally used; see e.g. [20] and [34].

3.2 Worst-Case Performance Problem

When dealing with the worst-case performance problem, we look for a probabilistic estimate of

$$u_{\max} = \max_{\Delta 2 \mathbf{\Delta}_\rho} u(\Delta).$$

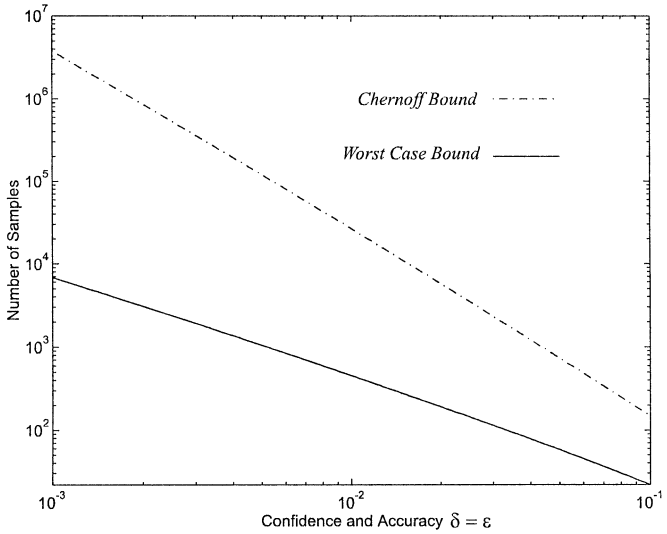

Figure 2: Comparison between Chernoff and worst-case bounds

To this end, we adopt again a random sampling scheme, as in the performance verification problem, generating N i.i.d. samples within $\mathbf{\Delta}_\rho$. Consequently, we obtain

$$\hat{u}_N \doteq \max_{i=1,2,\ldots,N} u(\Delta^i).$$

For this special formulation of the probabilistic robustness problem, a bound on the sample size N is given in [22] and [37].

Bound for worst-case performance
For any $\epsilon \in (0,1)$ *and* $\delta \in (0,1)$, *if*

$$N \geq \frac{\log \frac{1}{\delta}}{\log \frac{1}{1-\epsilon}} \tag{4}$$

then

$$\text{Prob}\{\text{Prob}\{u(\Delta) > \hat{u}_N\} \leq \epsilon\} \geq 1 - \delta. \tag{5}$$

In [15] the bound (4) is shown to be tight if the distribution function of Δ is continuous.

We remark that the worst-case performance problem is a special case of the general empirical probability estimation discussed in the previous subsection. In fact, setting $\gamma = \hat{u}_N$ in (2) we obtain

$$p_\gamma = \text{Prob}\,\{u(\Delta) \leq \hat{u}_N\} = 1 - \text{Prob}\,\{u(\Delta) > \hat{u}_N\}$$

and

$$\hat{p}_N = \frac{N_{good}}{N} = 1.$$

Therefore, equation (3) becomes

$$\text{Prob}\{|p_\gamma - \hat{p}_N| \leq \epsilon\} = \text{Prob}\{\text{Prob}\{u(\Delta) > \hat{u}_N\} \leq \epsilon\}.$$

Comparing the bound (4) with the Chernoff bound, it can be immediately verified that the number of samples required for the worst-case performance problem is much smaller than that needed for the performance verification problem; see Figure 2. However, there is no guarantee that $\hat{u}_N$ is actually close to the maximum $u_{\max}$. As shown in Figure 3, the bound previously discussed only guarantees that the set of points greater than the estimated value has a measure smaller than ϵ and this is true with a probability at least $1 - \delta$. In turn, this implies that if the function $u(\Delta)$ is sufficiently smooth, the estimated and actual maximum may be close.

In [2] a similar problem, with only one level of probability, instead of the two probability levels studied in this section, is considered. However, in this case, the sample size may be an exponential function of the number of uncertainties and therefore computational complexity becomes a critical issue. Finally, we recall that many results on sample complexity, based on the introduction of powerful concepts like VC-dimension (a measure of the problem complexity), may be found in [39]; see also [24] for some recent results.

4 Sample Generation Problem

As discussed in the previous section, a problem which is critical in the probabilistic setting is the development of efficient algorithms for sample generation in various sets according to several distributions. The interested reader may refer to [17] and [27] for a general discussion on the topic of random number generation. In particular, in [17] several algorithms for univariate sample generation according to various distributions are shown, while in [27] Monte Carlo and Quasi-Monte Carlo methods are analyzed in details. However, no specific algorithm for vector and matrix sample generation within sets of interest in robust control is

provided in the Monte Carlo literature. We also remark that standard rejection methods cannot be used for their inefficiency, see details in [13]. In the context of uncertain control systems described in the M-Δ form, the problem is the sample generation within $\mathbf{\Delta}_\rho$ according to a given density function f_Δ.

For real and complex parametric uncertainties $q_1, q_2, \ldots, q_\ell$, bounded in the ℓ_p norm-ball of radius ρ

$$\mathcal{B}_{\mathbb{F}}(\rho) \doteq \left\{ q \in \mathbb{F}^\ell : \|q\|_p \leq \rho \right\}, \tag{6}$$

the sample generation problem has a simple solution. We report here an algorithm, presented in [10], that returns a real random vector q uniformly distributed in the norm-ball $\mathcal{B}_{\mathbb{R}}(\rho)$. This algorithm is based on the so-called Generalized Gamma density $\bar{G}_{a,c}(x)$, defined as

$$\bar{G}_{a,c}(x) = \frac{c}{\Gamma(a)} x^{ca\,i\,1} \mathrm{e}^{i\,x^c}, \quad x \geq 0,$$

where a and c are given parameters and $\Gamma(a)$ is the Gamma function.

Algorithm for uniform sample generation in $\mathcal{B}_{\mathbb{R}}(\rho)$

1. Generate ℓ independent random real scalars ξ_i distributed according to $\bar{G}_{\frac{1}{p},p}(x)$.
2. Construct the random vector $x \in \mathbb{R}^\ell$ of components $x_i = s_i \xi_i$, where s_i are independent random signs. The random vector $y = \frac{x}{k x k_p}$ is uniformly distributed on the boundary of $\mathcal{B}_{\mathbb{R}}(\rho)$.
3. Generate $z = w^{1/\ell}$, where w is a random variable uniformly distributed in $[0,1]$. Return $q = \rho z y$.

Figure 4 visualizes the main steps of this algorithm in the simple case of sample generation of two dimensional real vectors in a circle of radius one ($\ell = 2$, $p = 2$, $\rho = 1$). First, we notice that for $p = 2$ the Generalized Gamma density $\bar{G}_{\frac{1}{p},p}(x)$ is related to the Gaussian density function. The random samples drawn from a Gaussian distribution (step 1) are radially symmetric with respect to the ℓ_2 norm. Roughly speaking, this means that their level curves are ℓ_2-spheres. Secondly, the samples are normalized obtaining random vectors uniformly distributed on the boundary of the circle (step 2), and then injected according to the volumetric factor z (step 3). We remark that in [10] a similar algorithm for complex vectors uniformly distributed in the norm-ball $\mathcal{B}_{\mathbb{C}}(\rho)$ is presented. The related concept of p-normality is discussed in [21] and an algorithm for the sample generation on the boundary of the ℓ_2 norm-ball is given in [26]

The sample generation problem becomes much harder when we are interested in the uniform generation of real and complex matrix samples $\Delta \in \mathbb{F}^{n,m}$ bounded in the induced ℓ_p-norm. In particular, while the cases $p = 1$ and $p = \infty$ can be immediately reduced to multiple random vector generation for which the techniques

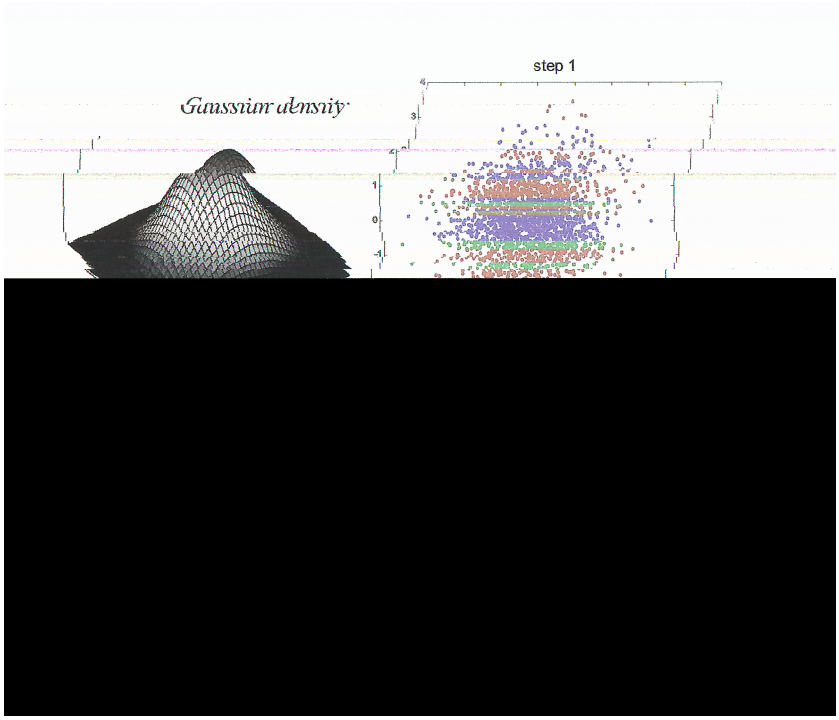

Figure 4: Generation of real random vectors uniformly distributed in a circle

described in [11] can be used, the solution for the induced ℓ_2-norm ball requires the development of specific methods. In particular, the algorithms presented in [13] and [14] are based on the singular value decomposition of the complex (real) matrix Δ in the matrix product

$$\Delta = U\Sigma V^{\star}$$

where U and V are unitary (orthogonal) matrices and Σ is a diagonal matrix containing the singular values of Δ. The main idea is to compute the density functions of U, Σ and V, respectively, such that the resulting pdf $f_{\Delta}(\Delta)$ is uniform. This approach is an extension of the methods described in [1] and [25] in the context of the theory of random matrices for the special case when Δ is a real symmetric matrix.

5 Probabilistic Robust LQ Regulators

In this section, we discuss probabilistic robust design of uncertain systems in the classical setting of Linear Quadratic Regulators. We consider a state space description of the M-Δ system given in Figure 1, where the structured uncertainty $\Delta \in \mathbf{\Delta}_{\rho}$ enters into the state matrix of the system

$$\dot{x}(t) = A(\Delta)x(t) + Bu(t), \tag{7}$$

with initial conditions $x(0) = x_0$.

The performance index is the standard quadratic cost function

$$J \doteq \int_0^{\infty} (x^T(t)Sx(t) + u^T(t)Ru(t))dt$$

where $S = S^T \geq 0$ and $R = R^T > 0$ are given weights. We take a state feedback law of the form

$$u(t) = -R^{-1}B^TQ^{-1}x(t) \tag{8}$$

with $Q = Q^T > 0$. For given $\gamma > 0$, it is well-known that the LQR problem can be reformulated in terms of the Quadratic Matrix Inequality (QMI)

$$A(\Delta)Q + QA^T(\Delta) - 2BR^{-1}B^T + \gamma(QSQ + BR^{-1}B^T) < 0.$$

That is, if a solution $Q = Q^T > 0$ of this QMI exists for all $\Delta \in \mathbf{\Delta}_\rho$, then the control law (8) quadratically stabilizes the system (7), and the cost

$$J \leq \gamma^{-1}x_0^TQ^{-1}x_0$$

is guaranteed for all $\Delta \in \mathbf{\Delta}_\rho$, see e.g. [30].

The solution given in [32] is based on a sequential algorithm. At each step of the sequence, the algorithm is based upon two stages

- Generation of a random sample Δ^k of the uncertainty $\Delta \in \mathbf{\Delta}_\rho$.
- Subgradient computation for the convex constraint defined by the QMI for Δ^k.

The first stage depends on the specific uncertainty structure under attention and can be performed using the methods described in Section 4. The outcome of this stage is a random sample $\Delta^k \in \mathbf{\Delta}_\rho$. The second stage requires finding an "approximate solution" of a convex constraint defined by the guaranteed cost problem. This can be immediately performed by subgradient step and projection on a cone of nonnegative definite matrices. The outcome of this stage is to obtain Q^k.

In [32], under fairly mild assumptions, it is shown that this sequential algorithm provides a controller which guarantees a given cost with probability one in a finite number of steps. In addition, at a fixed step k, a lower bound of the probability that a quadratically stabilizing controller is found is also computed. One of the advantages of this solution is that very general uncertainty structures can be easily handled. For example, when dealing with quadratic stability of an uncertain system affected by structured or nonlinear parametric uncertainty, standard methods based on Linear Matrix Inequalities (LMI) [9] cannot be used without introducing overbounding. In addition, even if this conservatism is acceptable, the LMI solution generally requires to simultaneously solve a number of convex inequalities which is exponential in the number of parameters. Unless the

problem size is very small, this issue is computationally critical so that finding a feasible solution with standard interior point methods may be very difficult or even intractable. On the other hand, the solution discussed here deals with only one constraint at each step of the sequence and therefore computational complexity is not an issue.

6 Conclusion

This paper is focused on nonstandard tools for analysis and control of uncertain systems, with emphasis on the interplay of probability and robustness. The goal is to combine hard bounds, which are frequently used in classical robust control, with probabilistic information which is often neglected in this context. The main advantage is to provide additional insight to the control engineer. This insight may be very useful in analyzing and designing uncertain control systems. We believe that future research will be mainly directed towards the syntesis of probabilistic robust controllers for output feedback and the development of probabilistic optimization algorithms.

References

[1] Anderson T.W. (1984) An Introduction to Multivariate Statistical Analysis. Wiley, New York

[2] Bai E.W., Tempo R. and Fu M. (1998) Worst Case Properties of the Uniform Distribution and Randomized Algorithms for Robustness Analysis. Mathematics of Control, Signals, and Systems **11**, 183–196

[3] Barmish B.R. (1994) New Tools for Robustness of Linear Systems. McMillan, New York

[4] Barmish B.R. and Lagoa C.M. (1997) The Uniform Distribution: A Rigorous Justification for its use in Robustness Analysis. Mathematics of Control, Signals, and Systems **10**, 203–222

[5] Bernoulli J. (1713) Ars Conjectandi

[6] Bhattacharyya S.P., Chapellat H. and Keel L.H. (1995) Robust Control: The Parametric Approach. Prentice-Hall, Englewood Cliffs

[7] Blondel V. and Tsitsiklis J.N. (1997) NP-Hardness of Some Linear Control Design Problems. SIAM Journal of Control and Optimization **35**, 2118–2127

[8] Blondel V. and Tsitsiklis J.N. (2000) A Survey of Computational Complexity Results in Systems and Control. Automatica **36**, 1249–1274

[9] Boyd S.P., El Ghaoui L., Feron E. and Balakrishnan V. (1994) Linear Matrix Inequalities in Systems and Control Theory. SIAM Publications, Philadelphia

[10] Calafiore G., Dabbene F. and Tempo R. (1998) Uniform Sample Generation in ℓ_p Balls for Probabilistic Robustness Analysis. Proceedings of the IEEE Conference on Decision and Control, Tampa, Florida

[11] Calafiore G., Dabbene F. and Tempo R. (1999) Radial and Uniform Distributions in Vector and Matrix Spaces for Probabilistic Robustness. Topics in Control and its Applications (eds. D. Miller and L. Qiu), Springer-Verlag, London, 17–31

[12] Calafiore G., Dabbene F. and Tempo R. (1999) The Probabilistic Real Stability Radius. Proceedings of the 14th World Congress of IFAC. Beijing, China

[13] Calafiore G., Dabbene F. and Tempo R. (2001) Randomized Algorithms for Probabilistic Robustness with Real and Complex Structured Uncertainty. IEEE Transactions on Automatic Control **46**, to appear

[14] Calafiore G. and Dabbene F. (2000) Polynomial-Time Random Generation of Uniform Real Matrices in the Spectral Norm Ball. Proceedings of IFAC Symposium on Robust Control Design, Prague, Czech Republic

[15] Chen X. and Zhou K. (1998) Order Statistics and Probabilistic Robust Control. Systems and Control Letters **35** 175–182

[16] Chernoff H. (1952) A Measure of Asymptotic Efficiency for Test of Hypothesis Based on the Sum of Observations. Annals of Mathematical Statistics **23** 493–507

[17] Devroye L. (1986) Non-Uniform Random Variate Generation. Springer-Verlag, New York

[18] Doyle J. (1982) Analysis of Feedback Systems with Structured Uncertainty. IEE Proceedings **129-D**, 242–250

[19] Fam A.T. (1989) The Volume of Coefficient Space Stability Domain of Monic Polynomials. Proceedings of the IEEE International Symposium on Circuits and Systems

[20] Fukunaga K. (1972) Introduction to Statistical Pattern Recognition. Academic Press, New York

[21] Gupta A.K. (1997) Characterization of p-Generalized Normality. Journal of Multivariate Analysis **60** 61–71

[22] Khargonekar P.P. and Tikku A. (1996) Randomized Algorithms for Robust Control Analysis Have Polynomial Time Complexity. Proceedings of the Conference on Decision and Control, Kobe, Japan

[23] Khatri S.H. and Parillo P.A. (1998) Spherical μ. Proceedings of the American Control Conference, Philadelphia

[24] Koltchinskii V., Abdallah C.T., Ariola M., Dorato P. and Panchenko D. (2001) Improved Sample Complexity Estimates for Statistical Learning Control of Uncertain Systems. IEEE Transactions on Automatic Control **46**, to appear

[25] Mehta M.L. (1991) Random Matrices. Academic Press, Boston

[26] Muller M.E. (1959) A Note on a Method for Generating Random Points Uniformly Distributed on n-Dimensional Spheres. Communications of the ACM **2**, 19–20

[27] Niederreiter H. (1992) Random Number Generation and Quasi-Monte Carlo Methods. SIAM Publications, Philadelphia

[28] Nemirovskii A. (1993) Several NP-Hard Problems Arising in Robust Stability Analysis. Mathematics of Control, Signals and Systems **6**, 99–105

[29] Papoulis A. (1965) Probability, Random Variables, and Stochastic Processes. McGraw-Hill, New York

[30] Petersen I.R. and McFarlane D.C. (1994) Optimal Guaranteed Cost Control and Filtering for Uncertain Linear Systems. IEEE Transactions on Automatic Control **39** 1971–1077

[31] Polyak B.T. and Shcherbakov P.S. (2000) Random Spherical Uncertainty in Estimation and Robustness. IEEE Transactions on Automatic Control **45**, to appear

[32] Polyak B.T. and Tempo R. (2000) Probabilistic Robust Design with Linear Quadratic Regulators. Proceedings of the IEEE Conference on Decision and Control, Sydney, Australia

[33] Qiu L., Bernhardsson B., Rantzer A., Davison E.J., Young P.M. and Doyle J.C. (1995) A Formula for Computation of the Real Stability Radius. Automatica **31** 879–890

[34] Ray L.R. and Stengel R.F. (1993) A Monte Carlo Approach to the Analysis of Control System Robustness. Automatica **29**, 229–236

[35] Safonov M.G. (1982) Stability Margins of Diagonally Perturbed Multivariable Systems. IEE Proceedings **129-D**, 251–256

[36] Stengel R.F. and Ray L.R. (1991) Stochastic Robustness of Linear Time-Invariant Control Systems. IEEE Transactions on Automatic Control **36**, 82–87

[37] Tempo R., Bai E.W. and Dabbene F. (1997) Probabilistic Robustness Analysis: Explicit Bounds for the Minimum Number of Samples. Systems and Control Letters **30**, 237–242

[38] Tempo R. and Dabbene F. (1999) Probabilistic Robustness Analysis and Design of Uncertain Systems. Dynamical Systems, Control, Coding, Computer Vision, (eds. G. Picci, D.S. Gilliam) Birkhäuser, Basel 263–282

[39] Vidyasagar M. (1997) A Theory of Learning and Generalization. Springer-Verlag, London

[40] Zhou K., Doyle J.C. and Glover K. (1996) Robust and Optimal Control. Prentice-Hall, Upper Saddle River

[41] Zhu X., Huang Y. and Doyle J. (1996) Soft vs. Hard Bounds in Probabilistic Robustness Analysis. Proceedings of the Conference on Decision and Control, Kobe, Japan